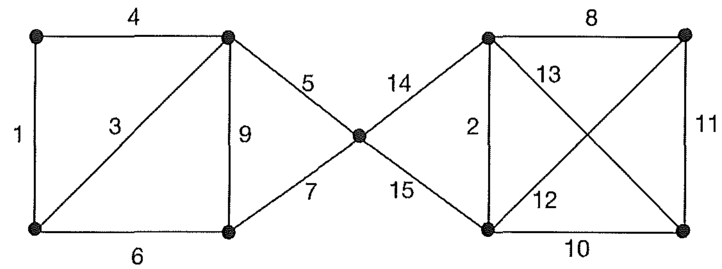
EXAM DISCRETE MATH

23 January 2023, 15:00-17:00

- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids
- Make sure to clearly and precisely state any results from the lecture notes you are using
- As usual, when using (exponential) generating functions you can ignore issues to do with the radius of convergence etc
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet** This is worth 10 points (out of a total of 100)

Exercise 1 (20 pts).
Let a_n satisfy the recursion $a_n - 4a_{n-1} + 4a_{n-2} = n$ for $n \geq 2$ and initial conditions $a_0 = a_1 = 1$ Determine an (explicit, closed form) expression for a_n

Exercise 2 (20 pts).
Apply Kruskal's algorithm to find a minimum spanning tree in the following edge-weighted graph



Make sure to clearly indicate, for each step of the algorithm, what actions are taken by the algorithm

(see next page)

Exercise 3 (20 pts)
Let $I_1, \ldots, I_n \subseteq \mathbb{R}$ be open intervals of length one, and let G be their intersection graph. That is $V(G) = [n]$ and $ij \in E(G)$ iff $I_i \cap I_j \neq \emptyset$. (Such a graph is called a *unit interval graph*.)
Show that $\chi(G) = \omega(G)$.

(*Hint*: consider "greedy" colouring from left to right.)

Exercise 4 (a:5,b:5,c:5,d:5,e:5,f:5 pts).
Recall that the Stirling numbers of the second kind, denoted $u_{n,k}$ in the lecture notes, count the number of unordered partitions of $[n]$ into precisely k non-empty parts.

a) Let $(a_n)_{n\geq 0}$ be a sequence of real numbers and let $(b_n)_{n\geq 0}$ be defined via

$$b_n = \sum_{k=0}^{n} u_{n,k} \cdot a_k \quad (\forall n \geq 0).$$

Show that the EGFs $\hat{A}, \hat{B}$ of these sequences satisfy the relation

$$\hat{B}(z) = \hat{A}\left(e^z - 1\right).$$

(*Hint*: You may wish to first recall, or derive from scratch, the expression for the EGF $\hat{U}_k$ of the sequence $(u_{u,k})_{n\geq 0}$ given in the lectures, and then see if you can rewrite the LHS or RHS.)

Recall that a cycle in a permutation π of $[n]$ is a set of distinct elements $i_1, \ldots, i_m \in [n]$ such that $\pi(i_1) = i_2, \ldots, \pi(i_{m-1}) = i_m, \pi(i_m) = i_1$. The Stirling numbers of the **first** kind $s_{n,k}$ count the number of permutations of $[n]$ with precisely k cycles. While it is debatable whether a "permutation of the empty set" makes any sense, we find it convenient to set $s_{0,0} = 1$ and $s_{0,k} = 0$ for all $k \geq 1$.

b) Show that $s_{n,k}$ satisfies the following recursion for $n, k \geq 1$:

$$s_{n,k} = s_{n-1,k-1} + (n-1) \cdot s_{n-1,k}.$$

(*Hint*: either $\pi(n) = n$ or not.)

c) Show that

$$\hat{S}_k(z) = \frac{1}{k!} \cdot \ln\left(\frac{1}{1-z}\right)^k,$$

for all $k \geq 0$.

(*Hint.* as a first step you could use the previous part to derive the differential equations $(1-z) \cdot \hat{S}'_k = \hat{S}_{k-1}$ for $k \geq 1$.)

d) As you might have already observed, we have the identity

$$\sum_k \hat{S}_k(z) = \frac{1}{1-z}.$$

Give a "combinatorial" explanation for this identity in terms of numbers of permutations.

(see next page)

e) Show that if $(a_n)_{n\geq 0}$ is some sequence and $(b_n)_{n\geq 0}$ is given by

$$b_n = \sum_k (-1)^{n-k} \, s_{n,k} \, a_k,$$

then the EGFs $\hat{A}, \hat{B}$ satisfy the identity

$$\hat{B}(z) = \hat{A}\left(\ln(1+z)\right).$$

(*Hint*: as an intermediate step you might want to determine the EGF of the numbers $(-1)^{n-k} \, s_{n,k}$, which are sometimes called the *signed* Stirling numbers of the first kind.)

f) Now deduce the following statement, known as "Stirling inversion": if $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are such that

$$b_n = \sum_k u_{n,k} \, a_k \quad (\forall n),$$

then

$$a_n = \sum_k (-1)^{n-k} \, s_{n,k} \, b_k \quad (\forall n)$$

(The end)